Quantization of Hitchin Systems

(1) Intro to Geometric Representation Theory

(2) A slice of Geometric Langlands Correspondence

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1. Borel-Weil Theorem

let G connected, semisimple Lie group$/\mathbb{C}$

<u>Ex</u> $SL_n = SL_n(\mathbb{C})$

Want to understand representations of G

$\rightsquigarrow$ Consider a flag manifold, $\mathcal{B}$ the set of Borel subgroups of G

<u>Ex</u> $\mathcal{B} = \{\text{upper triangular matrices}\}$

$G \curvearrowright \mathcal{B}$ by $g \cdot B = g \cdot B \cdot g^{-1}$

<u>Prop</u> (1) $G \curvearrowright \mathcal{B}$ transitive

(2) $N_G(B) = B$

$(\{g \in G : g \cdot B = B\} = B)$

$\Rightarrow \mathcal{B} = G/B$ algebraic variety

<u>Ex</u> $G = SL_2 \curvearrowright \mathbb{C}^2$

$B \leftrightarrow$ stabilizer of a line

$\mathcal{B} \cong \{\text{lines in } \mathbb{C}^2\} \cong \mathbb{P}^1$

G/B with $G = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1\}$, $B = \{\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}\}$

$\Rightarrow \{\begin{pmatrix} z_1 & * \\ z_2 & * \end{pmatrix} \mid \begin{matrix} (z_1, z_2) \neq 0 \\ (z_1, z_2) \sim \lambda(z_1, z_2) \end{matrix}\}$

Why $\mathcal{B}$?

① $G \rightsquigarrow \mathcal{B}$ proj variety

② Borel subgroups are important for Rep theory of G

③ Borel-Weil uses $\mathcal{B}$

④ Beilinson-Bernstein uses $\mathcal{B}$

Let $G \curvearrowright V$ irreducible f.d. rep'n

$$B \leftrightarrow l_B \subset V$$

$$\mathcal{B} \leftrightarrow \{l_B \subset V\}_{B \in \mathcal{B}}$$

$B \curvearrowright l_B$: 1-dim

$\text{ab} \downarrow$

$B/[B,B] =: H$

$[B,B] = N$ nilpotent radical

$G = SL_n \rightsquigarrow H = (\mathbb{C}^\times)^{n-1}$

$H \curvearrowright \{l_B\}_{B \in \mathcal{B}}$ $\quad \chi : H \to \mathbb{C}^\times$ character

<u>Defn</u> A G-equivariant vector bundle $E \to Y$, $G \curvearrowright Y$ is a vector-bundle $E \to Y$ together with

$$\sigma^* E = \pi_2^* E$$

where $\underset{\text{action}}{\sigma : G \times Y \to Y}$ $\quad \underset{\text{projection}}{\pi_2 : G \times Y \to Y}$

In particular $E_x \underset{\text{linear iso}}{\cong} E_{g \cdot x}$ $\quad \forall x \in Y,\ g \in G$

G - equiv vector bundles on Y

$\Leftrightarrow$ vector bundles on $G \backslash Y$

$\{G$-equiv line bundles on $G/B\}$

$\Leftrightarrow \{$line bundles on $pt/B\}$

$\Leftrightarrow \{$1-dim rep'ns of $B\}$

$\Leftrightarrow \{$1-dim rep'ns of $H\}$

$\Leftrightarrow \{$characters of $H\}$

$$\chi : H \to \mathbb{C}^\times \rightsquigarrow \underset{\mathcal{L}_\chi}{\overset{G\text{-equivariant}}{\Downarrow}} \text{ on } G/B$$

$$\mathcal{L}_{\chi,B} = \ell_B^*$$

G-equiv vector bundle E

For its section s

$(g \circ s)(x) = g s(g^{-1}x)$

$\Gamma(Y, E)$ is G-rep'n

<u>Thm</u> (Borel-Weil)

If λ is a dominant weight,

then $H^0(G, \mathcal{L}_\lambda)$ is rep'n with

Highest weight λ

<u>Fact</u> Any f.d. irrep of G is a highest weight.

$\Rightarrow$ Any such rep arises in this way

<u>Ex</u> $G = SL_2$, $\mathcal{B} = \mathbb{P}^1$

$\lambda = 0 \rightsquigarrow \mathcal{L}_0 = \mathbb{P}^1 \times \mathbb{C}$

$H^0(\mathcal{B}, \mathcal{L}_0) = \mathcal{O}(\mathbb{P}^1) = \mathbb{C}$

$\underset{n \geq 0}{\lambda = n} \rightsquigarrow \mathcal{L}_\lambda = \mathcal{O}(n)$

$H^0(\mathcal{B}, \mathcal{L}_\lambda) = H^0(\mathbb{P}^1, \mathcal{O}(n))$

deg n polys in x, y

<u>Rmk</u> For general λ,
one can describe $H^i(\mathcal{B}, \mathcal{L}_\lambda)$ ... due to Bott

(1) 2. Beilinson – Bernstein localization

rep'ns of $\mathfrak{g}$ of arbitrary, not f.d.

$\underset{\text{proj}}{H^0(\mathcal{B}, \mathcal{L}_\lambda)} \to$ f.d. $\Rightarrow$ can't just look at line bundle

Slogan: "Rep theory of $G \subseteq$ Geometry of $\mathcal{B}$"

Let $G \circlearrowright X$ smooth$/\mathbb{C}$

$\rightsquigarrow \mathfrak{g} \to \mathrm{Vect}(X)$

$\mathcal{U}(\mathfrak{g}) = \Gamma(X, \mathcal{D}_X) = \mathcal{D}(X)$

$[\Gamma(X, \mathcal{O}_X) = \mathcal{O}_X = \mathcal{O}(X)]$

$\mathcal{O}_X \subset \mathcal{D}(X)$

$\mathcal{D}_X$-mod $\xrightarrow{\Gamma} D_X(X)$-mod $\xrightarrow{\varphi^*} \mathcal{U}(\mathfrak{g})$-mod

Take $X = \mathcal{B}$

<u>Thm</u> Belinson - Bernstein

$\Phi : \mathcal{U}(\mathfrak{g}) \to \mathcal{D}(\mathcal{B})$ is surjective

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classical limit = associated graded

$$\begin{tikzcd} \mathcal{U}(\mathfrak{g}) \arrow[rr] & & \mathcal{D}(\mathcal{B}) \arrow[d, "Gr"] \\ Sym(\mathfrak{g}) = \mathcal{O}(\mathfrak{g}^*) \arrow[u, squiggly] \arrow[rr] & & \mathcal{O}(T^*\mathcal{B}) = \mathcal{O}(\mathcal{N}) \end{tikzcd}$$

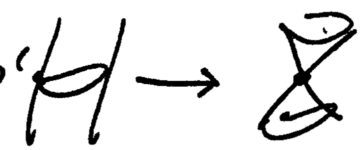
$\mathcal{N} \subset \mathfrak{g}^*$

$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ $a^2 + bc = 0$

$$\begin{tikzcd} T^*\mathcal{B} \arrow[rr, "\mu"] \arrow[dr, two heads] & & \mathfrak{g}^* \\ & \mathcal{N} \arrow[ur] & \end{tikzcd}$$

Show $\mathcal{O}(\mathfrak{g}^*) \twoheadrightarrow \mathcal{O}(\mathcal{N})$ surj
and $\mathcal{N} \hookrightarrow \mathfrak{g}^*$ $\Rightarrow$ done

<u>Rmk</u> $T^*\mathcal{B} \to \mathcal{N}$
Springer resolution

$\mathbb{P}^1$ 

Study of $T^*\mathcal{B} \to \mathcal{N}$ or its variants
= Springer ~~resolution~~
Theory

What is ker $\Phi : \mathcal{U}(\mathfrak{g}) \to \mathcal{D}(\mathcal{B})$

Consider $Z(\mathfrak{g}) = Z(\mathcal{U}(\mathfrak{g}))$

By Schur's lemma, for irrep V of $\mathfrak{g}$

$Z(\mathfrak{g}) \circlearrowright V$ as a scalar

$z . V = \chi(z) . V$ where $\chi : Z\mathfrak{g} \to \mathbb{C}$
central character

$Z\mathfrak{g} \simeq (\mathrm{Sym}\, \mathfrak{h})^W$ $\mathfrak{h} \subset \mathfrak{g}$

$\wr\mathfrak{S}$

$\mathbb{C}[x_1, \dots, x_r]$

where $x_1 \dots x_r$ are W-symmetric pol's

Cartan

W weyl group

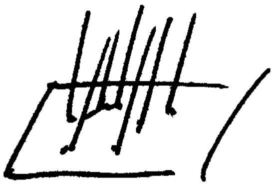
$r = \dim \mathfrak{h} = \mathrm{rk}\, \mathfrak{g}$

$\chi : Z\mathfrak{g} \to \mathbb{C}$

$\Rightarrow \chi \in \mathrm{Spec}(Z\mathfrak{g}) \cong \mathbb{A}^r$

Given rep'n of $\mathfrak{g}$

- first try to understand {irreps}
- try to understand how $Z\mathfrak{g}$ acts

Irrep $\mathfrak{g}$  / spec$(Z\mathfrak{g})$

$\Phi : U\mathfrak{g} \twoheadrightarrow \mathcal{D}(X)$

$\mathcal{D}(X)\text{-mod} \xrightarrow{\Phi^*} U(\mathfrak{g})\text{-mod}$

$\ker \Phi = U(\mathfrak{g}) \cdot \ker \chi_0$

$\Rightarrow \mathcal{D}(X)\text{-mod} \xrightarrow{\sim} U(\mathfrak{g})\text{-mod}_{\chi_0}$

where $Z\mathfrak{g}$ acts through χ_0

$\mathcal{D}^\lambda(X)\text{-mod} \xrightarrow{\sim} U\mathfrak{g}\text{-mod}_{\chi_\lambda}$

$\uparrow$

twisted diff. op.

$$\mathcal{D}_X\text{-mod} \xrightarrow{\Gamma} \mathcal{D}(X)\text{-mod} \xrightarrow[\simeq]{\Phi^*} U(\mathfrak{g})\text{-mod}_{\chi_0}$$

when $X = \mathcal{B} = G/B$

X variety

$$QCoh(X) \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} \mathcal{O}(X)\text{-mod}$$

localization for $X = \operatorname{spec} A$

$(\Delta M)(U_f) = M_f$

$U_f = \{f \neq 0\}$

$$\mathcal{D}_X\text{-mod} \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} D(X)\text{-mod}$$

$M \mapsto \Delta M = \mathcal{D} \otimes_{\mathcal{D}_X} M$

<u>Thm</u> (BB)

$X = \mathcal{B} = G/B$ is <u>D-affine</u>

$$\begin{tikzcd} U\mathfrak{g}\text{-mod}_{\chi_0} \arrow[rr, "\Delta"] & & \mathcal{D}_{\mathcal{B}}\text{-mod} \arrow[dl, "\Gamma", "\sim"'] \\ & \mathcal{D}(\mathcal{B})\text{-mod} \arrow[ul, "\Phi^*", "\sim"'] & \end{tikzcd}$$

$M \in U(\mathfrak{g})\text{-mod}$

$\Delta M = \mathcal{D} \otimes_{\mathcal{D}(X)} M$

Ex $G = SL_2$, $\mathcal{B} = \mathbb{P}^1$

$U_1 = \{[z_1, z_2] : z_2 \neq 0\}$ $\quad z = z_1/z_2 \quad$ on $U_1 \cap U_2$

$U_2 = \{[z_1, z_2] : z_1 \neq 0\}$ $\quad x = z_2/z_1 \quad$ $z = 1/x$

$$\widetilde{G} = \left\{ \begin{pmatrix} z_1 & * \\ z_2 & * \end{pmatrix} : \underset{\substack{\neq \\ 0}}{(z_1, z_2)} \sim \lambda(z_1, z_2) \right\}$$

$G \curvearrowright \widetilde{G}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x = \frac{a+bx}{c+dx}$$

$$\frac{d}{dz} = -x^2 \frac{d}{dx}$$

$$\mathrm{Vect}(\mathbb{P}^1) = H^0(\mathbb{P}^1, \mathcal{O}(2))$$
$$= \left\langle \frac{d}{dz}, z\frac{d}{dz}, z^2\frac{d}{dz} \right\rangle$$

$\mathfrak{g} \to \mathrm{Vect}(X)$

$sl_2 \to \mathrm{Vect}\,\mathbb{P}^1$ Lie alg. map

$$\left.\frac{d}{dt}\right|_{t=0} \exp(te_1)\varphi(z) = \left.\frac{d}{dt}\right|_{t=0} \varphi(e^{-te_1} z)$$

e, F, h $\Rightarrow$ $e \mapsto -\frac{d}{dz}$

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$f \mapsto z^2 \frac{d}{dz}$

$h \mapsto -2z\frac{d}{dz}$

$\mathcal{U}(sl_2) \twoheadrightarrow \mathcal{D}(X)$

$Z = \mathbb{C}(c)$, $c = eF + Fe + \frac{h^2}{2}$ $\Rightarrow \mathcal{U}\,sl_2 / \mathcal{U}\,sl_2 \cdot c \xrightarrow{\sim} \mathcal{D}(X)$

$c \mapsto 0$ $\qquad \mathcal{D}_X\text{-mod} \rightsquigarrow \mathcal{U}\,sl_2\text{-mod}_{\chi_0}$

$\mathbb{A}^1$

$$\begin{array}{ccccccccc}
 & & \mathcal{O}(\mathbb{A}^1) & & \mathcal{O}(\mathbb{A}^1\setminus\{0\}) & & \delta_0 & & \\
 & & \| & & \| & & \cap & & \\
0 & \to & \mathbb{C}[x] & \to & \mathbb{C}[x,x^{-1}] & \to & \mathbb{C}[x,x^{-1}]/\mathbb{C}[x] & \to & 0 \\
 & & \| & & \| & & \| & & \\
 & D/D\partial \cong & M_1 & D/D(\partial x) \cong & M_{1/x} & D/Dx \cong & M_{\delta_0} & & \\
 & & \Uparrow & & \uparrow & & \uparrow & & \\
 & & \{\partial\cdot 1=0\} & & \{\partial x\cdot 1/x=0\} & & \{x\cdot\delta_{p_0}=0\} & &
\end{array}$$

D-mod $\qquad\qquad$ Rep.

- $\mathcal{O}_{\mathbb{P}^1} \overset{\Gamma}{\rightsquigarrow} \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C} = L_0$

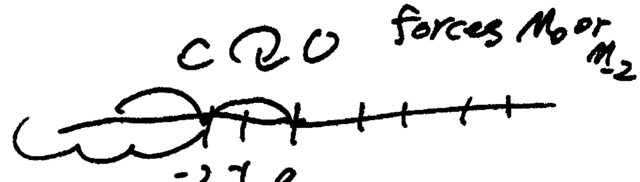
  simple module w/ h.w. 0

- $\mathbb{A}^1 \overset{j}{\hookrightarrow} \mathbb{P}^1$

  $j_*\mathcal{O}_{\mathbb{A}^1 = \mathbb{P}^1\setminus\{\infty\}} \rightsquigarrow \Gamma(\mathbb{P}^1, j_*\mathcal{O}_{\mathbb{A}^1}) = \mathbb{C}[x] = \mathbb{C}[z^{-1}]$

  $= M_0^\vee$

  dual verma module w/ h.w. 0



- $j_!\mathcal{O}_{\mathbb{A}^1} \rightsquigarrow\ = M_0$

  $\mathrm{im}(M_0 \to M_0^\vee) = L_0$

  $\mathrm{im}(j_! \to j_*) = j_{!*}$

$L_0 = M_0/M_{-2}$

$L_0 \subset M_0^\vee$

$M_{-2} = M_0^\vee/L_0$

- $j_*\mathcal{O}_{\mathbb{A}^1}/\mathcal{O}_{\mathbb{P}^1}$ only around 0

  $\cup$

  $j_*\mathcal{O}_{\mathbb{A}^1\setminus\{\infty\}}/\mathcal{O}_{\mathbb{P}^1\setminus\{\infty\}} = \delta_0 \rightsquigarrow \Gamma(\mathbb{P}^1, \delta_0) = \mathbb{C}[z^{-1}]/\mathbb{C}$

  $= M_{-2} = L_{-2}$

  $= M_{-2}^\vee = L_{-2}^\vee$

$\mathfrak{g}$-rep $\leftrightarrow$ (twisted) D-mod on $\mathcal{B}$

$(\mathfrak{g}, H)$-rep $\leftrightarrow$ H-equivariant D-mod on $\mathcal{B}$

$\mathrm{Lie}\, H \subset \mathfrak{g}$

$\mathrm{Fun}(K \backslash H / K)$

<u>Rmk</u> (Fate) $H = B$

$(\mathfrak{g}, B)$-rep $\leftrightarrow$ B-equiv D-mod on $\mathcal{B}$

$\updownarrow$ $\quad B \backslash G / B$

h.w. theory (category $\mathcal{O}$) $\leftrightarrow$ Schubert cells

<u>Rmk</u> $H = K \subset G$ compact

$\mathfrak{g} = \mathfrak{sl}_2$, $K = SO_2(\mathbb{C}) = \mathbb{C}^\times$

$\leftrightarrow$ $G_{\mathbb{R}}$ rep'n

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(2)

1. Fourier-Mckay transform for D-modules
and Geometric Langlands correspondence for $G = GL_1$

A abelian variety $\to$ $A^\vee$ dual abelian variety

$QC(A) \simeq QC(A^\vee)$

deg 0 line bundle $\mathcal{L}$ on A $\leftrightarrow$ $\mathcal{O}_{\mathcal{L}}$ skyscraper

$D(\mathrm{Bun}_G) \simeq ?$

$D(A) \simeq ?$

$\{A \to B\mathbb{G}_m\} \leftrightarrow A^\vee$

$\downarrow$

$QC(A)$

$A^\flat = \{\text{flat line bundles on } A\}$

<u>Thm</u> (Laumon, Rothstein)

$$D(A) = QC(A^\flat)$$

$(\mathcal{L}, \nabla)$ flat line bundle $\leftrightarrow \mathcal{O}_{(\mathcal{L},\nabla)}$ skyscraper

~~let $A = \operatorname{Jac} C$~~

$T^*A = A \times H^0(A, \Omega_A)$ $\qquad A^\vee \times H^0(A, \Omega_A)$

$A \cong \operatorname{Alb} A$ $\qquad H^0(A, \Omega_*)$

$$QC(A) \simeq QC(A^\vee \times H^0(A, \Omega_A))$$

$\rightsquigarrow$ def $\qquad \rightsquigarrow$ def

$$D(A) \simeq QC(A^\flat)$$

let $A = \operatorname{Jac} C \Rightarrow A^\flat = \{\text{Flat line bundles on } C\}$

$= \underline{\operatorname{Pic}^0} C \qquad = \underline{\operatorname{Flat}}_1 C$

$$\rightsquigarrow D(\underline{\operatorname{Pic}}^0 C) \simeq QC(\underline{\operatorname{Flat}}_1 C)$$

$\cup \qquad \cup$

$? \quad \xleftarrow[\text{loc}]{\simeq} \quad ?$

$\underline{\mathrm{Flat}}_1 C \xrightarrow{\pi} \mathrm{Jac}\, C$

$\pi^{-1}(\mathcal{O}_c) \to \mathcal{O}_c$

$\mathcal{E} = (\mathcal{O}_c, d+\omega)$ skyscraper on $\underline{\mathrm{Flat}}_1 C$

$\omega \in H^0(C, \Omega_C)$

$\rightsquigarrow (\mathcal{O}_{\mathrm{Jac}}, d+\tilde{\omega})$ flat line bundle on $\mathrm{Jac}\, C$

$\tilde{\omega} \in H_0(\mathrm{Jac}, \Omega_{\mathrm{Jac}})$

$\overset{\parallel}{\mathcal{F}_{\mathcal{E}}}$

$$H^0(C, \Omega_C) = H^0(\mathrm{Jac}, \Omega_{\mathrm{Jac}})$$
$$\omega \rightsquigarrow \tilde{\omega}$$

$$T\,\mathrm{Jac} = \mathrm{Jac} \times H^1(C, \mathcal{O}_C)$$
$$\mathrm{Vect}(\mathrm{Jac}) = H_0^1(C, \mathcal{O}_C)$$

$\xi = H^1(C, \mathcal{O}_C)$

$\nabla_\xi = \xi + \langle \omega, \xi \rangle$

flat connection on Jac

$$D_{\mathrm{Jac}} = \mathcal{D}(\mathrm{Jac}) = \mathrm{Sym}\, H^1(C, \mathcal{O}_C)$$
$$= \mathcal{O}(H^0(C, \Omega_C))$$

$$\lambda_\omega : D_{\mathrm{Jac}} \to \mathbb{C}$$
$$\xi \mapsto -\langle \xi, \omega \rangle$$

$$F_\mathcal{E} = \mathcal{D}/\ker \lambda_\omega = \mathcal{D} \otimes_D \mathbb{C}_\omega$$

Hitchin Fibration	$T^*Jac \to H^0(C,\Omega_C)$	$T^*Bun_G \to B$
Hitchin section	$H^0(C,\Omega_C) \subset T^*Jac$	$B \to HS \subset T^*Bun_G$
integrability	$\mathcal{O}(B) = \mathcal{O}(T^*Jac)$	$\mathcal{O}(B) = \mathcal{O}(HS)$ $= \mathcal{O}($
quantization	$\mathcal{O}(B) = D_{Jac}$ $\quad \| \quad$ $U(H^1(C,\mathcal{O}_C))$	$\mathcal{O}(\mathcal{O}p_S) = H_0(C,\mathfrak{g})$

	Jac	Bun_G
$U(\mathfrak{g}) \to D_X$-mod $M \mapsto D \otimes_{U\mathfrak{g}} M$	$M \mapsto D \otimes_{D_{Jac}} M$ $D_{Jac} = U(H^1(C,\mathcal{O}))$	$M \mapsto D \otimes_D M$